The Future of Tilted Bianchi Universes

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Abstract

An asymptotic stability analysis of spatially homogeneous models of Bianchi type containing tilted perfect fluids is performed. Using the known attractors for the non-tilted Bianchi type universes, we check whether they are stable against perturbations with respect to tilted perfect fluids. We perform the analysis for all Bianchi class B models and the Bianchi type VI_0 model. In particular, we find that none of the non-tilted equilibrium points are stable against tilted perfect fluids stiffer than radiation. We also indicate parts of the phase space where new tilted exact solutions might be found.

1 Introduction

In order to explain the isotropy and homogeneity of the universe and investigate the form of its small expansion anisotropies, it is important to understand the complete spectrum of anisotropic cosmological models. The subset of the most general anisotropic and inhomogeneous cosmological models that are most amenable to a full analysis are the spatially homogeneous Bianchi universes because the Einstein field equations for these cosmologies reduce to ordinary differential equations. They have been studied in considerable detail by many authors but attention has largely focussed on the behaviour of models with comoving perfect fluids. In this paper we will extend these analyses to the situation where the fluid is 'tilted' with respect to the normals to the hypersurfaces of homogeneity and the fluid is not comoving.

In the study of Bianchi type universes there is special interest in the particular exact solutions of the Einstein equations which are attractors for more

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general classes of solution (see [1] for a review). The two recent papers, [2, 3], have completed the investigation of all non-chaotic Bianchi models with a *non-tilted* γ-law perfect fluid and their asymptotic attractors are understood. This understanding takes the form of a phase-space description of the dynamics of the autonomous ordinary differential equations describing the cosmological evolution. However, when we turn to Bianchi models with *tilted* fluid motion [4], the situation is far from completely understood. Up to three additional degrees of freedom can be added to the phase spaces in these cases. Some analyses have been made of Bianchi universes with tilted fluids [5, 6], in particular type II [7] and type V [8, 9, 10], but a more general analysis has not been by date. The object of this paper is to provide a local stability analysis of a wide range of Bianchi universes containing tilted perfect fluids in order to elucidate their late-time behaviours. Specifically, we will consider Bianchi class B models, together with the type VI_0 model of class A, filled with a tilted γ-law perfect fluid. We will base our analysis on the results from the non-tilted analysis. The main reason for this is that, due to the complexity of the equations of motion, a complete analysis would need to find, or rule out the existence, of new tilted exact solutions which might act as attractors. However, a relatively simple analysis based on the results from the non-tilted models allows us to go a significant way towards understanding the behaviour of tilted universes.

Tilted universes display many features that are of considerable interest in the study of Einstein's equations and their physical implication. They include rotating universes and are therefore of importance for any attempts to formulate and evaluate Mach's Principle [11], the conditions needed for the appearance of closed timelike curves, and the properties of quantum cosmologies that tunnel from 'nothing'. The changes in the cosmological evolution of anisotropies that arise in brane-worlds are also of special interest but, so far, the influence of non-comoving velocities have not been studied. They are likely to be the most important factor in the evolution as can be seen from the effects of anisotropic pressures studied by Barrow and Maartens [12].

Before we embark on the formal analysis of the full Einstein equations governing the tilted Bianchi universes it is helpful to use some elementary physical arguments to determine when we might expect critical changes in the behaviour of velocities and vorticities to occur as the equation of state varies. Using a fairly simple argument we can give a rough estimate of the stability properties to be expected of non-comoving perfect fluids [6]. A perfect fluid with equation of state, $p = (\gamma - 1)\rho$, and constant γ, will evolve according to $\rho \propto a^{-3\gamma} \propto \theta^2$, where $a(t)$ is the mean scale factor and $\theta = 3\dot{a}/a$ is the volume expansion rate. Consider an expanding universe that is close to isotropy: an eddy of fluid, with mass m, will conserve angular momentum, $I\omega$, if

$$Ma^2\omega = \text{constant}.$$

For tilted fluids with velocity $v = a\omega$ and mass $m \propto \rho a^3$, this implies

$$\rho a^5\omega = \rho a^4 v = \text{constant}.$$

Thus for perfect fluids

$$v \propto a^{3\gamma-4}, \quad \omega \propto a^{3\gamma-5}.$$

The ratio of the distorting *rotational energy density* ω^2 to the isotropic density evolves as

$$\frac{\omega^2}{\theta^2} \propto \frac{\omega^2}{\rho} \propto a^{9\gamma-10}.$$

We see that we expect particular values of γ to mark thresholds in the evolution of models containing non-comoving velocities and vorticity. In particular a threshold exists when $\gamma = 4/3$: rotational velocities grow with $a(t)$ for $\gamma > 4/3$ [13]. The influence of vorticity on the expansion dynamics grows when $\gamma > 10/9$. The significance of the value $\gamma = 10/9$ can be seen, for example, in the exceptional model, discussed section 3.5. In the analyses to follow we find that these critical values of γ recur, along with the value $\gamma = 2/3$, which marks a transition between universes that isotropise at late times (for $\gamma < 2/3$) and those that do not (for $\gamma > 2/3$). Of course, the detailed behaviour is more complicated than this discussion implies because the Bianchi types need not be close to isotropy and there are geometrical constraints on how the different components of the 3-velocity are able to evolve if energy and momentum are to be conserved [1].

In the next section we will determine the equations of motion for the tilted fluid in Bianchi universes. Then, in section 3, we will perform a stability analysis for the various asymptotic behaviours that are known from the study of non-tilted universes. In section 4 we will analyse the type VI_h model. Finally, in section 5 we summarise and discuss our results.

2 The equations of motion

We choose a time variable so that the time is measured along time-like geodesics orthogonal to the surfaces of transitivity. The surfaces of transitivity are therefore spatial and we say that the spacetime is *spatially homogeneous*. For the Bianchi cosmologies – which admit a simply transitive symmetry group acting on the spatial hypersurfaces – we can always write the line-element as

$$ds^2 = -dt^2 + g_{ab}(t)\boldsymbol{\omega}^a\boldsymbol{\omega}^b,$$

where g_{ab} is a non-singular symmetric 3×3 matrix only dependent upon time, $\boldsymbol{\omega}^a$ is a triad of one-forms obeying

$$\mathbf{d}\boldsymbol{\omega}^a = -\frac{1}{2}C^a{}_{bc}\boldsymbol{\omega}^b \wedge \boldsymbol{\omega}^c,$$

[1] This simple physical argument can be extended to the early stages of brane-world cosmologies. In brane-worlds we have $\theta \propto \rho$ rather than $\theta^2 \propto \rho$ at early times and so the distortion to the metric created by any vortical motions is much more significant than in general relativistic cosmologies as the universe expands. We have $\omega/H \propto a^{6\gamma-5}$ and rotational distortions grow in all brane-worlds with $\gamma > 5/6$ and this includes the cases of matter and radiation domination.

and $C^a{}_{bc}$ are the structure constants of the Bianchi group type under consideration. The structure constants $C^a{}_{bc}$ can be split into a vector part a_b, and a trace-free part n^{ab} by [14]

$$C^a{}_{bc} = \varepsilon_{bcd} n^{da} - \delta^a{}_b a_c + \delta^a{}_c a_b.$$

The matrix n^{ab} is symmetric, and, using the Jacobi identity, $a_b = (1/2)C^a{}_{ba}$ is in the kernel of n^{ab}

$$n^{ab} a_b = 0.$$

The Bianchi types are divided into two classes:

- Class A: $a_c = 0$.
- Class B: $a_c \neq 0$.

First, we will concentrate on the class B spacetimes. In this case we can choose the orientation of the triad so that $a_c = a\delta^1{}_c$. This implies that n^{ab} is of rank 2 and can be written

$$\left(n^{ab}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & n^{22} & n^{23} \\ 0 & n^{23} & n^{33} \end{bmatrix}.$$

Consider a perfect fluid with energy-momentum tensor

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}.$$

The normalised 4-velocity of the fluid u^μ is in general tilted with respect to the normals to the hypersurfaces of homogeneity, $n_a = t_{,a}$; and the fluid three-velocity v^a, where $u^\mu = (u^0, v^a)$, will be assumed to be non-zero. The equations of motion for the fluid, $T^{\mu\nu}{}_{;\nu} = 0$, can be written as a closed system of equations [15,16]

$$\begin{aligned} -(\rho + p)\dot{v}_a v^a &= \dot{p} v_a v^a & (1) \\ -(\rho + p)\left(u^0 \dot{v}_c + v_a v^b C^a{}_{bc}\right) &= \dot{p} u^0 v_c & (2) \\ (\rho + p)\left(\dot{u}^0 + v^b a_b + u^0 \theta\right) + u^0 \dot{\rho} &= 0. & (3) \end{aligned}$$

Here, θ is the volume expansion factor $\theta = \dot{g}/2g$. In our analysis, eqs. (1-3) are the essential equations because we will consider the stability of non-tilted solutions, that obey the Einstein equations with $v_a \equiv 0$, with respect to the perturbations introduced by the present of non-comoving velocities.

Recently, the two remaining Bianchi models with non-tilted γ-law perfect fluid have been analysed [2,3]; so all non-tilted Bianchi models have now been analysed and their asymptotic behaviours determined. Using these results, which give all the attractors in the set of non-tilted perfect fluid cosmologies, we can investigate the stability of the late-time non-tilted attractors against perturbations from tilted matter.

It is useful to introduce a orthonormal triad $\mathsf{e}^{\hat{a}}{}_{a}(t)$ by defining

$$\delta_{\hat{a}\hat{b}}\mathsf{e}^{\hat{a}}{}_{a}\mathsf{e}^{\hat{b}}{}_{b} = g_{ab}, \quad \delta_{\hat{a}\hat{b}} = \mathsf{e}^{a}{}_{\hat{a}}\mathsf{e}^{b}{}_{\hat{b}}g_{ab}.$$

We can now write eqs. (1-3) in terms of the components $v_{\hat{a}} = \mathsf{e}^{a}{}_{\hat{a}}v_a$ in the orthonormal frame:

$$\begin{aligned} -(\rho+p)\left(\dot{v}_{\hat{a}}v^{\hat{a}} + \dot{\mathsf{e}}^{\hat{a}}{}_{a}\mathsf{e}^{a}{}_{\hat{b}}v_{\hat{a}}v^{\hat{b}}\right) &= \dot{p}v_{\hat{a}}v^{\hat{a}} \quad (4)\\ -(\rho+p)\left(u^0\dot{v}_{\hat{c}} + u^0\mathsf{e}^{a}{}_{\hat{c}}\dot{\mathsf{e}}^{\hat{b}}{}_{a}v_{\hat{b}} + v_{\hat{a}}v^{\hat{b}}\mathsf{e}^{\hat{a}}{}_{a}\mathsf{e}^{b}{}_{\hat{b}}C^{a}{}_{bc}\mathsf{e}^{c}{}_{\hat{c}}\right) &= \dot{p}u^0v_{\hat{c}} \quad (5)\\ (\rho+p)\left(\dot{u}^0 + v^{\hat{b}}\mathsf{e}^{b}{}_{\hat{b}}a_b + u^0\theta\right) + u^0\dot{\rho} &= 0. \quad (6) \end{aligned}$$

The stability of a particular solution can now be checked by linearising these equations around the exact non-tilted solution describing the asymptote in question. For non-tilted equilibrium points, the linearised version of eqs. (4-5) will be a closed set of equations, and hence, the stability analysis can to a large extent be done by considering these equations alone (provided the stability analysis has already been performed for the remaining equations).

More precisely, our stability analysis is based on the following observation. The total dynamical system (including the Einstein equations) can be written close to an equilibrium point, say at $\mathbf{x} = 0$,

$$\dot{\mathbf{x}} = \mathsf{A}\mathbf{x} + \mathbf{F}(\mathbf{x}),$$

where $\mathbf{F}(\mathbf{x})$ is at least quadratic in $\mathbf{x}$. In our case, when considering a non-tilted equilibrium point, the matrix A will be of the block form

$$\mathsf{A} = \begin{bmatrix} \mathsf{A}_1 & 0 \\ \mathsf{B} & \mathsf{A}_2 \end{bmatrix}. \quad (7)$$

This means that the eigenvalues of A are the union of the eigenvalues of A_1 and A_2. Thus if we choose equilibrium points that are stable in the set of non-tilted cosmologies, we only have to check the eigenvalues for A_1; i.e. the linearised version of eq. (5). This simplifies the stability analysis considerably and turns out to be very fruitful. However, there are some limits to this procedure:

1. If the real parts of one or more eigenvalues of A_1 are zero, then the stability analysis is inconclusive from analysing A_1 alone.
2. For tilted equilibrium points, A will in general not have the form (7); i.e. the complete set of equations is needed to establish its stability.

Keeping these limitations in mind we shall perform a perturbation analysis for various spacetimes.

3 Stability analysis

We will study the various class B cases, plus the type VI_0 case which is closely related. We will henceforth assume that the tilted perfect fluid obeys the barotropic equation of state[2]

$$p = (\gamma - 1)\rho.$$

Note that ever-expanding cosmologies with inflationary types of fluids with $0 \leq \gamma < 2/3$, will all asymptote to a FRW model with $a(t) \propto t^{2/3\gamma}$ as $t \to \infty$ in accord with the cosmological "no-hair" theorem [19, 1]. These fluids will therefore not be considered explicitly in what follows. However, the $\gamma = 2/3$ will appear as a stability boundary for the anisotropic attractors as expected.

The notion of tilt does not apply to an exact vacuum solution. Hence, for equilibrium points corresponding to vacuum spacetimes we can only define a tilt when they are perturbed away from any exact vacuum solutions. Thus, in what follows, when refer to stability with respect to the introduction of tilt we are determining whether the universe at late times is best described by a tilted or a non-tilted model. Also, when referring to the vacuum solutions we will always mean the non-tilted equilibrium points.

3.1 Bianchi type VII_h

The most general Bianchi universes containing the open Friedmann universe as a subcase are those of type VII_h. The general late-time asymptotes for non-tilted spacetimes of type VII_h are the plane-wave vacuum solutions of Lukash [20].[3] For these solutions [21]

$$\begin{aligned} \boldsymbol{\omega}^1 &= \mathbf{dx} && (8)\\ \boldsymbol{\omega}^2 &= e^{sx}\left[\cos\omega x\mathbf{dy} - \sin\omega x\mathbf{dz}\right] && (9)\\ \boldsymbol{\omega}^3 &= e^{sx}\left[\sin\omega x\mathbf{dy} + \cos\omega x\mathbf{dz}\right] && (10)\\ s(1-s) &= \omega^2\sinh^2 2\beta, && (11) \end{aligned}$$

and

$$\mathsf{e}^{\hat{a}}{}_{a} = \begin{bmatrix} t & 0 & 0 \\ 0 & t^s e^{\beta}\cos(\omega\ln t) & -t^s e^{\beta}\sin(\omega\ln t) \\ 0 & t^s e^{-\beta}\sin(\omega\ln t) & t^s e^{-\beta}\cos(\omega\ln t) \end{bmatrix}$$

The group parameter, h, is given by

$$h = \frac{s^2}{\omega^2}.$$

[2]Non-perfect fluid stresses, for example anisotropic pressures, can have significant effects on the evolution of anisotropies; see for example refs. [17, 12].

[3]These vacuum solutions can easily be generalised to plane-wave solutions with an electromagnetic field [21].

Furthermore, $\theta = (2s+1)/t$, and they reduce to isotropy when $s = 1$ and $\omega = \beta = 0$. Eq. (6) is to lowest order in $v_{\hat{a}}$

$$\dot{\rho} + \gamma\theta\rho = 0. \tag{12}$$

This is the same as for non-tilted fluid. Defining expansion-normalised energy-density by $\Omega = \rho/\theta^2$, we require $\dot{\Omega} < 0$ for the solutions to be future stable. This yields the bound

$$\frac{2}{1+2s} < \gamma.$$

This is the usual bound required for the stability of these solutions in the space of non-tilted VII_h universes with perfect fluid found earlier [1, 18, 6]. Linearising eq. (5), using eq. (12), and dividing by $\gamma\rho$, we get the following system of equations for the 3-velocities:

$$\begin{aligned}
v'_{\hat{1}} &= [(2s+1)\gamma - (2s+2)]\, v_{\hat{1}} \\
v'_{\hat{2}} &= [(2s+1)\gamma - (3s+1)]\, v_{\hat{2}} - e^{2\beta}\omega v_{\hat{3}} \\
v'_{\hat{3}} &= [(2s+1)\gamma - (3s+1)]\, v_{\hat{3}} + e^{-2\beta}\omega v_{\hat{2}}.
\end{aligned} \tag{13}$$

Here, we have also introduced a new time-variable $\tau = \ln t$ and a prime denotes derivative with respect to τ. The eigenvalues of this system of equations, which determine the stability of the plane-wave solutions, are

$$\begin{aligned}
\lambda_1 &= (2s+1)\gamma - (2s+2), \\
\lambda_{2,3} &= (2s+1)\gamma - (3s+1) \pm i\omega.
\end{aligned} \tag{14}$$

Hence, the plane-wave solutions of type VII_h are stable when

$$\frac{2}{2s+1} < \gamma < \frac{3s+1}{2s+1}.$$

Note that $(3s+1)/(2s+1) \leq 4/3$; so the solutions are unstable for fluids stiffer than radiation. This reflects the tendency for very stiff fluids to 'spin up' as they expand in order to conserve angular momentum [13], as discussed above.

For radiation, $\gamma = 4/3$, the solutions are unstable for $s < 1$. The case $s = 1$ yields a zero eigenvalue for radiation and has to be considered separately. However, $s = 1$ implies that the solutions are LRS and are therefore the same as the LRS type V solutions. Type V evolution has been considered elsewhere, and, in particular, the stability of this solution has been analysed in detail [8, 9, 10]. For *generic* perturbations[4] this solution is found to be *unstable*. Hence, all plane-wave solutions of type VII_h are unstable for radiation ($\gamma = 4/3$).

Since for a given s, any value of h is possible, we can conclude that for any given h there are plane-wave solutions of type VII_h that are stable in the set of VII_h models, whenever

$$\frac{2}{3} < \gamma < \frac{4}{3}.$$

A summary of the results for type VII_h is illustrated in Fig. 1.

[4] In [10], it is shown that this solution is, in fact, an attractor for a set of non-zero measure of solution trajectories in the state space of irrotational type V models. However, for some perturbations it is unstable and is therefore generically unstable.

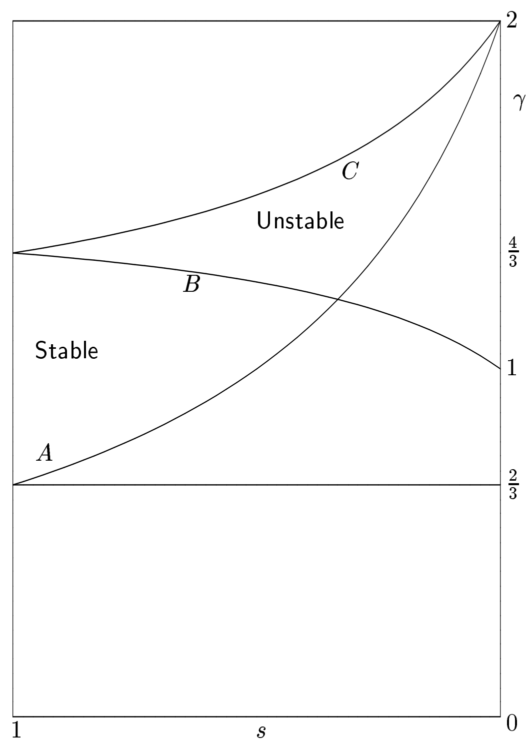

Figure 1: Stability diagram for the Bianchi type VII_h Lukash plane-wave solutions. Here, the labelled curves are defined by the following functions: A is $\gamma = \frac{2}{2s+1}$, B is $\gamma = \frac{3s+1}{2s+1}$ and C is $\gamma = \frac{2s+2}{2s+1}$. The region marked "Stable" is bounded by the curves A and B, while the region marked "Unstable" is bounded by A, B and C as shown. The latter region has two directions in which the solutions are unstable against tilt. All the remaining unmarked regions of the diagram are also those in which the Lukash solutions are unstable.

3.2 Bianchi type V

The general late-time attractor for non-tilted type V models is the isotropic Milne universe. This vacuum solution can be extracted from the Lukash plane waves above by choosing $\omega = 0$, $s = 1$. Hence, from the above we see that this solution is stable provided that

$$\frac{2}{3} < \gamma < \frac{4}{3}.$$

This agrees with earlier results of other authors [6, 8, 9, 10].

3.3 Bianchi type VI_h

Consider the type VI_h plane-wave solutions given by

$$\begin{aligned} \boldsymbol{\omega}^1 &= \mathbf{dx} && (15)\\ \boldsymbol{\omega}^2 &= e^{(s+b)x}\mathbf{dy} && (16)\\ \boldsymbol{\omega}^3 &= e^{(s-b)x}\mathbf{dz} && (17)\\ s(1-s) &= b^2 + r^2, && (18) \end{aligned}$$

and

$$\mathsf{e}^{\hat{a}}{}_a = \begin{bmatrix} t & 0 & 0 \\ 0 & t^{s+b} & -\frac{r}{b}t^{s-b} \\ 0 & 0 & t^{s-b} \end{bmatrix}.$$

The group parameter is given by

$$h = -\frac{s^2}{b^2},$$

and $\theta = (2s+1)/t$. Following the same procedure as in the type VII_h case, we obtain a lower bound from eq. (6):

$$\frac{2}{1+2s} < \gamma. \tag{19}$$

Linearising eq. (5) we get

$$\begin{aligned} v'_{\hat{1}} &= [(2s+1)\gamma - (2s+2)]\, v_{\hat{1}}, \\ v'_{\hat{2}} &= [(2s+1)\gamma - (3s+1+b)]\, v_{\hat{2}}, \\ v'_{\hat{3}} &= [(2s+1)\gamma - (3s+1-b)]\, v_{\hat{3}}, \end{aligned} \tag{20}$$

and the eigenvalues are:

$$\begin{aligned} \lambda_1 &= (2s+1)\gamma - (2s+2), \\ \lambda_{2,3} &= (2s+1)\gamma - (3s+1 \pm b). \end{aligned} \tag{21}$$

Assuming that bound (19) holds, we have

$$\lambda_3 > 2 - (3s+1-b) \geq \frac{3}{2}(1 - \sqrt{-h}).$$

Hence, all models with $|h| < 1$ are unstable. The threshold model, of LRS type $III = VI_{-1}$, has one solution with a zero eigenvalue.

Combining these results, we see that for $|h| > 1$ there are stable plane wave solutions whenever

$$\frac{2(1-h)}{1-3h} < \gamma < \frac{1-\sqrt{-h}-4h}{1-3h} < \frac{4}{3}.$$

Consider next the Collins VI_h perfect fluid solution

$$\begin{aligned} \boldsymbol{\omega}^1 &= \mathbf{dx} && (22) \\ \boldsymbol{\omega}^2 &= e^{\frac{r(2-\gamma)+c}{2\gamma}x}\mathbf{dy} && (23) \\ \boldsymbol{\omega}^3 &= e^{\frac{r(2-\gamma)-c}{2\gamma}x}\mathbf{dz} && (24) \\ c^2 &= (2-\gamma)(3\gamma-2), && (25) \end{aligned}$$

and

$$\mathsf{e}^{\hat{a}}{}_a = \begin{bmatrix} t & 0 & 0 \\ 0 & t^{\frac{2-\gamma+rc}{2\gamma}} & 0 \\ 0 & 0 & t^{\frac{2-\gamma-rc}{2\gamma}} \end{bmatrix},$$

where $0 < r < 1$, and

$$h = -\frac{(2-\gamma)^2 r^2}{c^2}.$$

This solution is stable to the future for non-tilted perfect-fluid cosmologies whenever

$$\frac{2}{3} < \gamma < \frac{2(1-h)}{1-3h}. \tag{26}$$

Linearising eq. (5), we get

$$\begin{aligned} v'_{\hat{1}} &= \frac{\gamma-2}{\gamma} v_{\hat{1}} \\ v'_{\hat{2}} &= \frac{1}{2\gamma}\left(5\gamma-6-rc\right) v_{\hat{2}} \\ v'_{\hat{3}} &= \frac{1}{2\gamma}\left(5\gamma-6+rc\right) v_{\hat{3}}. \end{aligned} \tag{27}$$

We note that

$$\lambda_1 < 0, \quad \lambda_2 < \lambda_3$$

for $\gamma < 2$ and so stability is assured if

$$\lambda_3 < 0 \Rightarrow -h < \frac{(5\gamma-6)^2}{(3\gamma-2)^2};$$

or in terms of γ,

$$\gamma < \frac{2(3+\sqrt{-h})}{5+3\sqrt{-h}}.$$

When $-h < 1$, this bound in tighter than the upper bound in eq. (26); when $-h > 1$ the bound (26) is the tightest. Hence, we can conclude that there are future stable Collins type VI_h perfect fluid solutions in the presence of tilt so long as

$$\frac{2}{3} < \gamma < \min\left(\frac{2(1-h)}{1-3h}, \frac{2(3+\sqrt{-h})}{5+3\sqrt{-h}}\right) < \frac{6}{5}.$$

3.4 Bianchi type IV

The late-time attractors for non-tilted type IV models are also plane waves. Their stability can be obtained from the type VI_h analysis by taking the limit $b \to 0$ to obtain the eigenvalues

$$\begin{aligned} \lambda_1 &= (2s+1)\gamma - (2s+2), \\ \lambda_{2,3} &= (2s+1)\gamma - (3s+1). \end{aligned} \tag{28}$$

Hence, there will be stable solutions whenever

$$\frac{2}{3} < \gamma < \frac{4}{3}.$$

3.5 The exceptional model of Bianchi type $VI^*_{-1/9}$

The asymptotic dynamics of the exceptional model has been studied with a non-tilted perfect fluid by Hewitt [22], and more recently by Horwood *et al* [2]. A stability analysis was done for both non-tilted fluid and with fluids with a single non-zero tilted component by Barrow and Sonoda [6].

Consider the Robinson-Trautman vacuum solution

$$\boldsymbol{\omega}^1 = \mathbf{dx} \tag{29}$$

$$\boldsymbol{\omega}^2 = e^{\frac{\sqrt{6}}{5}x}\mathbf{dy} \tag{30}$$

$$\boldsymbol{\omega}^3 = e^{-\frac{2\sqrt{6}}{5}x}\mathbf{dz} \tag{31}$$

and

$$\mathsf{e}^{\hat{a}}{}_a = \begin{bmatrix} t & 0 & 0 \\ \frac{\sqrt{5}}{2}t & t^{\frac{1}{5}} & 0 \\ 0 & 0 & t^{\frac{3}{5}} \end{bmatrix}.$$

This solution has unusual volume expansion, with $\theta = 9/(5t)$. This solution is an attractor for all non-tilted perfect fluids with $10/9 \leq \gamma$. This requirement for stability is easily seen to come from eq. (12). Linearising eq. (5), we obtain

$$\begin{aligned} v'_{\hat{1}} &= \frac{1}{5}(9\gamma - 14)v_{\hat{1}} \\ v'_{\hat{2}} &= \frac{1}{5}(9\gamma - 10)v_{\hat{2}} \\ v'_{\hat{3}} &= \frac{3}{5}(3\gamma - 4)v_{\hat{3}}. \end{aligned} \tag{32}$$

However, due to the exact vanishing of the 02-equation of Einstein's field equations, we have to set $v_{\hat{2}} = 0$. Hence, the type $VI^*_{-1/9}$ implies that only $v_{\hat{1}}$ and $v_{\hat{3}}$ can be non-zero. Thus we have only two eigenvalues to consider

$$\lambda_1 = \frac{1}{5}(9\gamma - 14), \quad \lambda_3 = \frac{3}{5}(3\gamma - 4).$$

The first of these eigenvalues was found by Barrow and Sonoda [6]. However, λ_3 gives an even tighter bound for stability. Thus, the Robinson-Trautman solution is stable if

$$\frac{10}{9} \leq \gamma < \frac{4}{3}.$$

For $\gamma = 10/9$, the attractor for the non-tilted case is the Wainwright $VI^*_{-1/9}$ $\gamma = 10/9$ perfect fluid solution [23] which interpolates between the Collins perfect-fluid solutions, studied earlier, and the Robinson-Trautman solution. The solution is

$$\boldsymbol{\omega}^1 = \mathbf{dx} \tag{33}$$

$$\boldsymbol{\omega}^2 = e^{\frac{r\sqrt{6}}{5}x}\mathbf{dy} \tag{34}$$

$$\boldsymbol{\omega}^3 = e^{-\frac{2r\sqrt{6}}{5}x}\mathbf{dz} \tag{35}$$

and

$$\mathsf{e}^{\hat{a}}{}_a = \begin{bmatrix} t & 0 & 0 \\ wt & t^{\frac{1}{5}} & 0 \\ 0 & 0 & t^{\frac{3}{5}} \end{bmatrix}.$$

Here, we have $\theta = 9/(5t)$, and $w^2 = 9r^2/4 - 1$.

Linearising eq. (5), we get the eigenvalues

$$\lambda_1 = -\frac{4}{5}, \quad \lambda_3 = -\frac{2}{5}.$$

Since both are negative, this solution set is asymptotically stable as well.

When $\gamma < 10/9$, the non-tilted solutions asymptote to the Collins $VI_{-1/9}$ perfect-fluid solution studied earlier. However, here we must again exclude one of the eigenvalues because of the vanishing of the (02) Einstein equation. Nonetheless, all eigenvalues are negative when $h = -1/9$ and $\gamma < 10/9$, so this does not tighten the bounds already arising in the non-tilted analysis. Hence, there are stable solutions of the exceptional model when

$$\frac{2}{3} < \gamma < \frac{10}{9}.$$

3.6 Bianchi type VI_0

This model can be obtained as the $h \to 0$ limit of type VI_h universes, both geometrically and dynamically. In the non-tilted case, the Collins VI_0 perfect-fluid solution is the future attractor for $2/3 < \gamma < 2$. Allowing for tilted fluids

we see that the Collins VI_0 perfect-fluid solution is stable whenever

$$\frac{2}{3} < \gamma < \frac{6}{5}.$$

4 Tilted Bianchi type VI_h equilibrium points

An interesting point emerges from the analysis of the type VI_h models. We can see that the instability of the Collins perfect-fluid solution in the exceptional model with $\gamma = 10/9$ corresponds to an instability of the Collins solutions with respect to a tilted fluid for more general h (see Fig. 2, where this is illustrated by the fact that the line B intersects $h = -1/9$ at $\gamma = 10/9$). Hence, at $h = -1/9$, the instability with respect to tilt is exchanged for an instability of the shear: when $h = -1/9$ there does not need to be a matter source to drive the solution away from the Collins solutions. This may indicate that the exceptional model, in the presence of tilted fluid, is not as dynamically exceptional as one might think and that the rest of the type VI_h models will behave in a similar way. Thus there should exist exact tilted solutions of type VI_h, analogous to the Robinson-Trautman vacuum solutions, which act as attractors. Such solutions are not known in general, but we expect them to exist.

Hence, one is tempted to search for solutions on the form

$$\boldsymbol{\omega}^1 = \mathbf{dx} \tag{36}$$

$$\boldsymbol{\omega}^2 = e^{(s-b)x}\mathbf{dy} \tag{37}$$

$$\boldsymbol{\omega}^3 = e^{(s+b)x}\mathbf{dz} \tag{38}$$

and

$$\mathsf{e}^{\hat{a}}{}_{a} = \begin{bmatrix} t & 0 & 0 \\ w_1 t & t^{\sigma} & 0 \\ w_2 t & 0 & t^{\tau} \end{bmatrix},$$

where $w_{1,2}, \sigma, \tau, s$ and b are constants. Due to the power-law dependence of the triad, this metric is self-similar by construction and the Einstein equations reduce to purely algebraic equations. However, the equations are still quite complex and difficult to solve in full generality. Rosquist [24], and Rosquist and Jantzen [25] have found some solutions of Bianchi type VI_0 of this type. In general, such spacetimes will describe solutions with tilt and non-zero vorticity. Furthermore, due to the complexity of the solutions no analysis of their stability has been done. Preliminary results indicate that the solutions found by Rosquist and Jantzen may play an important role in the late-time behaviour for at least a class of tilted type VI_0 models [26]. Notwithstanding the lack of a complete analysis, we believe that these kinds of solutions may be important for the late-time behaviour of models in the remaining region in Fig. 2; namely that bounded by the curves $\gamma = 4/3$, A, and B. One piece of supporting evidence for this conjecture is, as already mentioned, the fact that the exceptional case intersects this region. We believe that the Robinson-Trautman solution shares

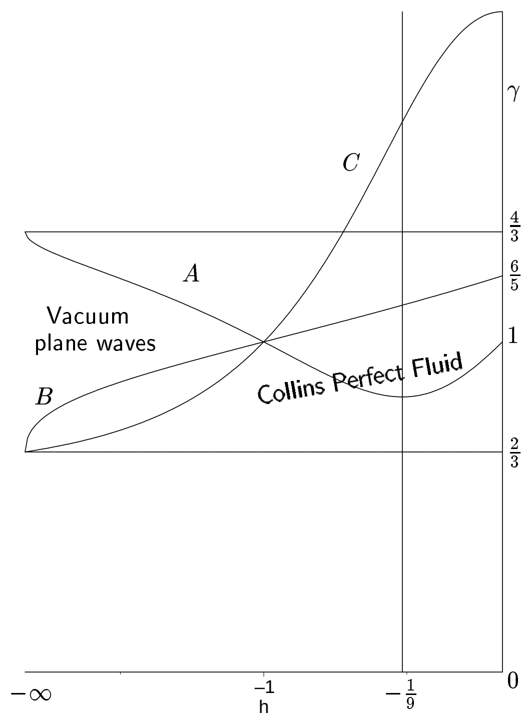

Figure 2: Stability diagram showing the stable solutions for the Bianchi type VI_h cases. The labelled curves are defined by the following functions: A is $\gamma = \frac{1-\sqrt{-h}-4h}{1-3h}$, B is $\gamma = \frac{2(3+\sqrt{-h})}{5+3\sqrt{-h}}$ and C is $\gamma = \frac{2(1-h)}{1-3h}$. The region marked "Vacuum plane waves" is bounded by the curves A and C, while the region marked "Collins perfect fluid" is bounded by $\gamma = 2/3$, B and C. The region bounded by $\gamma = 4/3$, A and B contains none of the solutions we have investigated as attractors, except for the exceptional case which cuts through this region at $h = -1/9$. For non-tilted cosmologies the region corresponding to "Collins perfect fluid" is the entire region bounded by $\gamma = 2/3$ and C; while for "Vacuum plane waves" it is the region above C.

many common features with these "missing" solutions (apart from the fact that this is the only vacuum solution).

5 Discussion

A summary of our results are given in table 1.

We have studied the stability of all the exact non-tilted class B and type VI_0 Bianchi-type universes which act as attractors for the general non-tilted cosmological evolution at late times. This has led to a determination of their local stability in the presence of non-comoving motions fluid motions. None of the non-tilted attractors are stable with respect to perfect fluids stiffer that radiation ($p > \rho/3$). The type VI_h models show a fairly complex dependence on the equation of state and the group parameter defining the 3-curvature anisotropy of the universe. However, there may be indications from this analysis that there are some "missing" stable models of this type which have not yet been found. The exceptional $VI^*_{-1/9}$ model is the only case with known exact solutions of this type. Further analysis of this "missing" set of solutions is required. For type VII_h, the situation is simpler: whenever $2/3 < \gamma < 4/3$ there are always some Lukash plane-wave metrics that are stable for any given $h \neq 0$.

This investigation has been a step towards acquiring a complete understanding of all tilted Bianchi models. Clearly, there are some missing pieces in the analysis: class A universes require a separate analysis and amongst those of class B the most prominent lacuna is the "missing" set of solutions of type VI_h. Also, we have only done a local stability analysis of non-tilted equilibrium point. The type V analysis [10] shows that the evolution in the whole state space may be far more complex than local perturbative analyses can reveal. New equilibrium points with non-zero tilt could exist which act as additional attractors. Furthermore, due to the nature of our method, the stability of the solutions along the boundary between stable and unstable regions (i.e. those that have one or more eigenvalues with zero real parts) remains unsettled. These models require further analysis. The Bianchi type III universes appear interesting in this context. The type III dust model ($\gamma = 1$) is the centre model in Fig. 2 where the curves A, B and C all intersect. Hence, the neighbouring models appear to have a very sensitive dependence on the defining parameters.

This study is has been a first attempt to extend our systematic understanding of the dynamics of Bianchi type universes to those with tilted fluid motions. It should be viewed as a first step towards gaining that understanding. We hope that the lacunae we have defined and the stability analyses we have presented will provide a basis for the development of a complete picture of the behaviour of tilted Bianchi type universes. Although non-comoving fluid motions have generally been neglected in studies of anisotropic models of the early universe, they are by no means negligible [28]. Under a wide range of realistic equations of state they have been shown to be the most important source of anisotropy in expanding universes. In the case of brane-world cosmologies [27] the effects of these motions will be even more dominant and no reliable understanding of the

Solution Type	Matter	Stability
VII_h (Lukash)	$\frac{2}{3}<\gamma<\frac{4}{3}$	Stable
	$\frac{4}{3}\leq\gamma\leq 2$	Unstable
IV (p-wave)	$\frac{2}{3}<\gamma<\frac{4}{3}$	Stable
	$\frac{4}{3}\leq\gamma\leq 2$	Unstable
V (Milne)	$\frac{2}{3}<\gamma<\frac{4}{3}$	Stable
	$\frac{4}{3}\leq\gamma\leq 2$	Unstable
VI_h (Collins)	$\frac{2}{3}<\gamma<\min\left(\frac{2(1-h)}{1-3h},\frac{2(3+\sqrt{-h})}{5+3\sqrt{-h}}\right)$	Stable
	$\frac{2(3+\sqrt{-h})}{5+3\sqrt{-h}}<\gamma<\frac{2(1-h)}{1-3h}$	Unstable
VI_h (p-wave)	$\frac{2(1-h)}{1-3h}<\gamma<\frac{1-\sqrt{-h}-4h}{1-3h}$	Stable
	$\frac{1-\sqrt{-h}-4h}{1-3h}<\gamma\leq 2$	Unstable
$\mathrm{VI}^*_{-1/9}$ (Collins)	$\frac{2}{3}<\gamma<\frac{10}{9}$	Stable
$\mathrm{VI}^*_{-1/9}$ (W)	$\gamma=\frac{10}{9}$	Stable
$\mathrm{VI}^*_{-1/9}$ (R-T)	$\frac{10}{9}<\gamma<\frac{4}{3}$	Stable
	$\frac{4}{3}<\gamma\leq 2$	Unstable
VI_0 (Collins)	$\frac{2}{3}<\gamma<\frac{6}{5}$	Stable
	$\frac{6}{5}<\gamma\leq 2$	Unstable

Table 1: A summary of our results: the exact solutions listed in the first column are all attractors in the non-tilted case for the equation of state and model parameters indicated in second column. The third column records whether they are stable or unstable when tilted fluids are introduced. Here, W=Wainwright, and R-T=Robinson-Trautman.

early evolution of these cosmologies will be possible in general without a full analysis of tilted fluid motions and vorticity.

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